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LETTER TO THE EDITOR

Supersymmetry breaking for dynamical systems

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Abstract. The Langevin dynamics of a non-potential system is considered. As a consequence the system is not invariant under supersymmetric transformations. Using Ward identities we demonstrate that there occur corrections to the linear fluctuation-dissipation theorem. Supersymmetry is also broken for stochastic systems driven by coloured noise.

It was realised [1,2] that there exists a hidden supersymmetry (SUSY) in certain Langevin dynamics. In this letter we consider two cases in which the SUSY is broken. As the consequence we discuss deviations from the usual fluctuation-dissipation theorem (FDT).

To be specific, the dynamics of the random process should be characterised by the Langevin equation

$$\dot{\varphi}_\alpha(t) = K_\alpha(\varphi) + \xi_\alpha(t) \tag{1}$$

where the label α denotes the space coordinate and an n-component vector index. The drift vector $K_\alpha(\varphi)$ is in general a nonlinear functional of the fields $\varphi_\alpha(t)$. The stochastic forces $\xi_\alpha(t)$ are coupled in an additive manner to the deterministic part. Here we investigate two cases of Gaussian processes (white noise and coloured noise) with zero mean and the correlators

$$\langle \xi_\alpha(t)\xi_\beta(t')\rangle = 2D_{\alpha\beta}\delta(t-t') \tag{2a}$$

or

$$\langle \xi_\alpha(t)\xi_\beta(t')\rangle = \frac{D_{\alpha\beta}}{\tau}\,\mathrm{e}^{-|t-t'|/\tau}. \tag{2b}$$

The diffusion matrix $D_{\alpha\beta}$ is symmetric and positive. In calculating the correlation function $C_{\alpha\beta}(t)$ and the response function $G_{\alpha\beta}(t)$ one has to resort to perturbation theory. In such an analysis the corresponding equations for C and G are coupled. In particular, there exist classes of models, discussed in [3], for which the two functions C and G are related linearly by the FDT. For instance, one finds for a potential system with Gaussian white noise (2a) and $D_{\alpha\beta}$ proportional to $T\delta_{\alpha\beta}$ (T is temperature)

$$C_{\alpha\beta}(\omega) = \frac{2T}{\omega}\,\mathrm{Im}\,G_{\alpha\beta}(\omega). \tag{3}$$

Such a FDT can be derived using a Ward–Takahashi identity [4-6].

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The FDT is not valid in the form (3) for non-equilibrium steady states where detailed balance does not apply. One aim of the present letter is the investigation of a model for which the potential conditions introduced in [3, 7] are not fulfilled and consequently the random process defined by (1) and (2a) does not obey detailed balance. The deterministic force is given by [8]

$$K_\alpha = -\Gamma \frac{\partial H}{\partial \varphi_\alpha(t)} + \Gamma \tau_{\alpha\beta}\varphi_\beta(t) \tag{4}$$

with a random matrix $\tau_{\alpha\beta}$ (in the continuous version $\tau_{\alpha\beta}$ means $\tau_{\alpha\beta}(\boldsymbol{x})$ with α and β as matrix indices running from l to n). The static random matrix is coupled to the fields φ_α in a multiplicative manner. In general, $\tau_{\alpha\beta}$ is not a symmetric matrix. For simplicity we assume a Gaussian white noise behaviour

$$\overline{\tau_{\alpha\beta}(\boldsymbol{x})\tau_{\gamma\delta}(\boldsymbol{x}')} = F_{\alpha\beta\gamma\delta}\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{5}$$

where the fourth-rank tensor $F_{\alpha\beta\gamma\delta}$ is specified for the concrete model by [8]

$$F_{\alpha\beta\gamma\delta} = 2[f\delta_{\alpha\gamma}\delta_{\beta\delta} + g\delta_{\alpha\delta}\delta_{\beta\gamma}] = 2f[\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}] - 2\varepsilon\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{6}$$

with $\varepsilon = f - g$.

The model defined by (1), (2a) and (4)-(6) is discussed in connection with Ising spin glasses [9], neural networks with asymmetric bonds [10] and an n-component vector model [8, 11] with $D_{\alpha\beta} = T\Gamma\delta_{\alpha\beta}$ where T is the temperature if the system reaches thermal equilibrium; otherwise, T measures the level of the dynamical stochastic noise; Γ is a kinetic coefficient. The investigation is restricted to the large-N case (N is the total number of Ising spins) or to the $n \to \infty$ limit since in these cases the dynamical mean field approximation can be applied [9]. By indicating the class of the diagrams the relation of the dynamical mean field approximation to a perturbational analysis is established in [11].

It is easy to show that the potential conditions (see [3]) are only fulfilled if the matrix $\tau_{\alpha\beta}(\boldsymbol{x})$ is symmetric. Denoting by $a_{\alpha\beta}$ the symmetric part of $\tau_{\alpha\beta}$ the deterministic force (4) can be rewritten as a potential system with the effective Hamiltonian $H' = H - \frac{1}{2}\int a_{\alpha\beta}\varphi_\alpha\varphi_\beta \, d^dx$. To make this point clearer we derive the stochastically equivalent Fokker-Plank equation for the conditional probability density $W(\varphi_\alpha, t; \tau_{\alpha\beta})$ using a standard procedure (see [12])

$$\Gamma^{-1}\frac{\partial W}{\partial t} = \frac{\partial}{\partial \varphi_\alpha}\left[\left(\frac{\partial H}{\partial \varphi_\alpha} + T\frac{\partial}{\partial \varphi_\alpha} - \tau_{\alpha\beta}\varphi_\beta\right)W\right]. \tag{7}$$

Equation (7) processes a stationary solution $W_{st} \propto e^{-H_{eff}/T}$ only if $\tau_{\alpha\beta}$ is a symmetric matrix with $H_{eff} = H'$. In terms of the new parameters f and g it means $f = g$. The parameter $\varepsilon = f - g \neq 0$ characterises the violation of detailed balance conditions.

Now, we demonstrate that the last fact is also manifested in a breakdown of supersymmetric transformations. To this aim we introduce the path-integral measure governed by the stochastic process (1) and (2a)

$$P[\varphi_\alpha; \xi_\alpha, \tau_{\alpha\beta}]\mathscr{D}\varphi = \delta(\dot{\varphi}_\alpha - K_\alpha - \xi_\alpha)|\mathscr{D}\xi/\mathscr{D}\varphi|\mathscr{D}\varphi. \tag{8}$$

The path probability density P depends on the set of parameters given here by $\xi_\alpha(\boldsymbol{x}, t)$ and $\tau_{\alpha\beta}(\boldsymbol{x})$. It fulfils the normalisation condition $\int P\mathscr{D}\varphi = 1$. Due to the multiplicative coupling (4) of the stochastic matrix $\tau_{\alpha\beta}(\boldsymbol{x})$ to the fields φ_α the Jacobian $|\mathscr{D}\xi/\mathscr{D}\varphi|$ becomes configuration dependent. Therefore it is more appropriate to express the Jacobian by functional integrals over Grassmann variables $\psi_\alpha(\boldsymbol{x}, t)$ and $\bar{\psi}_\alpha(\boldsymbol{x}, t)$/[1, 2].

Such an approach is different from that of [9]. The advantage of our procedure is that the introduction of these anticommuting fields allows, at least for potential systems, a superfield formulation.

Using an integral representation for the δ-functional in (8) by means of auxiliary (commuting) Bose fields $B_\alpha(x, t)$ and performing the Gaussian average over the dynamical noise $\xi_\alpha(x, t)$ we get

$$\langle P(\varphi_\alpha; \xi_\alpha, \tau_{\alpha\beta})\rangle \equiv P(\varphi_\alpha; \tau_{\alpha\beta}) = \int \mathscr{D}B\, D\bar{\psi}\, D\psi \exp(-S) \tag{9a}$$

with the action

$$S[\varphi, B, \bar{\psi}, \psi] = (B_\alpha|D_{\alpha\beta}D_\beta) + (\mathrm{i}B_\alpha|\dot{\varphi}_\alpha - K_\alpha) + (\bar{\psi}_\alpha|[\delta_{\alpha\beta}\partial_\alpha - \partial K_\alpha/\partial\varphi_\beta]\psi_\beta) \tag{9b}$$

and the notation for the scalar product

$$(a_\alpha|b_\alpha) = \sum_\alpha \int \mathrm{d}t\, \mathrm{d}x\, a_\alpha(x, t) b_\alpha(x, t). \tag{9c}$$

We remark that using (9a), performing integration over the Bose fields one can derive the generalised Fokker-Planck equation proposed recently by Graham [13].

In the special case where K_α is given by (4) the action S can be divided into two parts $S = S_0 + S_\tau$

$$S_0 = T\Gamma(B_\alpha|B_\alpha) + (\mathrm{i}B_\alpha|\dot{\varphi}_\alpha - \Gamma\partial H/\partial\varphi_\alpha) + (\bar{\psi}_\alpha|\dot{\psi}_\alpha) + \Gamma(\bar{\psi}_\alpha|(\partial^2 H/\partial\varphi_\alpha\partial\varphi_\beta)\psi_\beta) \tag{10a}$$

$$S_\tau = -\Gamma(\bar{\psi}_\alpha|\tau_{\alpha\beta}\psi_\beta) - \Gamma(\mathrm{i}B_\alpha|\tau_{\alpha\beta}\varphi_\beta). \tag{10b}$$

The first part, S_0, originating from the potential system $-\Gamma\partial H/\partial\varphi_\alpha$, can be rewritten in terms of a superfield

$$\Phi_\alpha(x, t, \bar{\Theta}, \Theta) = \varphi_\alpha(x, t) + \bar{\Theta}\psi_\alpha(x, t) + \bar{\psi}_\alpha(x, t)\Theta - \bar{\Theta}\Theta\mathrm{i}B_\alpha(x, t) \tag{11}$$

with new supersymmetric coordinates $\bar{\Theta}$ and Θ as elements of a Grassmann algebra. The part S_0 is invariant under a supersymmetric transformation [2, 5, 6]: $\Theta \to \Theta + \lambda$, $\bar{\Theta} \to \bar{\Theta} + \lambda$, $t \to t + \delta t$ with $\delta t = c_1\bar{\Theta}\lambda + c_2\bar{\lambda}\Theta$ where λ and $\bar{\lambda}$ are infinitesimal anticommuting parameters and the constants c_i are chosen to be $c_1 = 1/T\Gamma$ and $c_2 = 0$ [4]. The transformation of the 'coordinates' t, Θ and $\bar{\Theta}$ results in a transformation of the corresponding components:

$$\begin{aligned} \delta\bar{\psi}_\alpha &= -\bar{\lambda}\mathrm{i}B_\alpha + c_2\dot{\varphi}_\alpha\bar{\lambda} \qquad & \delta\psi_\alpha &= c_1\dot{\varphi}_\alpha\lambda - \mathrm{i}B_\alpha\lambda \\ \delta\varphi_\alpha &= \bar{\lambda}\psi_\alpha + \bar{\psi}_\alpha\lambda \qquad & \delta(\mathrm{i}B_\alpha) &= c_1\dot{\psi}\alpha\lambda - c_2\dot{\bar{\psi}}_\alpha\bar{\lambda}. \end{aligned} \tag{12}$$

The τ-dependent part of the action S_τ is only invariant under the supersymmetric transformation (12) if $\tau_{\alpha\beta} = \tau_{\beta\alpha}$ independent on the realisation of the F-tensor in (6). As a consequence of the broken supersymmetry the linear FDT (3) is not valid. Indeed, as pointed out in [5] the FDT is related to supersymmetry. Applying the invariance of S_0 by the transformation (12) one can derive the FDT from Ward-Takahashi identities.

In the case considered here we define the generating functional Z by the relation

$$Z(j, m, \bar{m}, \mathrm{i}I) = \int \mathscr{D}\varphi\{\overline{P[\varphi_\alpha; \tau_{\alpha\beta}]} \exp[(j_\alpha|\mathrm{i}B_\alpha) + (\bar{\psi}_\alpha|m_\alpha) + (\bar{m}_\alpha|\psi_\alpha) + (\mathrm{i}I_\alpha|\varphi_\alpha)]\} \tag{13}$$

where j_α, m_α, $\bar{m}_\alpha$ and $\mathrm{i}I_\alpha$ are the sources for the Bose fields $\mathrm{i}B_\alpha$, the Grassmann variables $\bar{\psi}_\alpha$ and ψ_α and the order parameter field φ_α, respectively. It is important to note that we take the structural averaged probability $\overline{P[\varphi_\alpha; \tau_{\alpha\beta}]}$ since we are interested in the structural averaged correlation function and the corresponding response function.

Due to the Gaussian white noise for the static disorder introduced in (6) the configurational average can be performed (see [9]) and it results in

$$P(\varphi_\alpha;\, \tau_{\alpha\beta}) \equiv P(\varphi_\alpha) = \exp[-\tilde{S}(\varphi_\alpha,\, B_\alpha,\, \bar{\psi}_\alpha,\, \psi_\alpha)]$$

with $\tilde{S} = S_0 + S_1$:

$$S_1 = -\Gamma^2 F_{\alpha\beta\gamma\delta}\{(\bar{\psi}_\alpha\psi_\beta \| \bar{\psi}_\gamma\psi_\delta) + (\mathrm{i}B_\alpha\varphi_\beta \| \mathrm{i}B_\gamma\varphi_\delta) + 2(\bar{\psi}_\alpha\psi_\beta \| \mathrm{i}B_\gamma\varphi_\delta)\} \tag{14}$$

and the notation $(a_\alpha \| b_\alpha) = \int \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}x\, a_\alpha(x, t) b_\alpha(x, t')$.

To derive the FDT we consider the behaviour of $\tilde{S}$ under the transformation (12) and find $\delta\tilde{S} = \delta S_0 + \delta S_1$ with $\delta S_0 = 0$ and

$$\delta S_1 = 2\varepsilon\Gamma^2\{(\bar{\psi}_\alpha \mathrm{i}B_\beta \| \mathrm{i}B_\alpha\varphi_\beta) - (\mathrm{i}B_\alpha\bar{\psi}_\alpha \| \mathrm{i}B_\alpha\varphi_\beta) + (\bar{\psi}_\alpha\psi_\beta \| \psi_\alpha \mathrm{i}B_\beta) - (\bar{\psi}_\beta\psi_\alpha \| \bar{\psi}_\alpha \mathrm{i}B_\beta)\}\lambda. \tag{15}$$

Following the procedure from reference [5] we get, after a straightforward calculation,

$$\frac{\mathrm{d}}{\mathrm{d}t} C_{\alpha\beta}(t-t') = T[G_{\alpha\beta}(t'-t) - G_{\alpha\beta}(t-t')] + R_{\alpha\beta}(t-t'). \tag{16}$$

Equation (16) is a generalised FDT. The additional $r_{\alpha\beta}$ is proportional to ε. In the case of vanishing ε the last relation is reduced after Fourier transformation to (3). The term $R_{\alpha\beta}$ can be represented by higher over correlation functions but it seems not to be possible to derive explicit corrections to (3) directly from (16).

For this reason we have used a conventional perturbation theory [11, 14] to find out nonlinear correction terms to the FDT (3) in terms of the parameter ε. Since there is no linear FDT we have to calculate the structural averaged correlation function and the corresponding response function separately. Following our previous analysis [11] we find $C_{\alpha\beta}(\omega) = G_{\alpha\gamma}(\omega)\Lambda_{\gamma\delta}(\omega)G_{\delta\beta}(-\omega)$, where the renormalised noise vertex part Λ [14, 16] is given through a perturbational series in terms of the parameter f and ε (6) and the nonlinearities in the Hamiltonian (we use an isotropic φ^4-model).

For non-zero ε the perturbational series for $C(\omega)$ is compared with that for $\mathrm{Im}\, G(\omega)$. After a straightforward calculation (for details see [15]) we get up to order ε^2 for small ε:

$$\begin{aligned} C(\omega) = 2T\, \mathrm{Im}\, G(\omega)/\omega \bigg\{ 1 + \varepsilon \frac{2\Gamma(n-1)}{\omega} \mathrm{Im}\, G(\omega)(1-Q) \\ + \varepsilon^2 \frac{2\Gamma(n-1)}{\omega} \mathrm{Im}\, G(\omega)\left[2|G(\omega)|^2 + \frac{2\Gamma(n-1)}{\omega} \mathrm{Im}\, G(1-Q)^2 \right]\bigg\} \end{aligned} \tag{17}$$

with $Q = 2f[G(\omega) + G(-\omega)]^2$.

Equation (17) yields nonlinear corrections to the FDT in the form (3). It is obvious that the corrections disappear for vanishing ε or $n = 1$ as indicated by (17).

As a further example for an altered FDT compared with (3) and supersymmetry breaking we consider a potential system driven by coloured Gaussian noise (2*b*). Using the observation [17] that the noise probability weight implied by (2*b*) is

$$P[\{\xi(t)\}] \sim \exp\left\{ -\frac{1}{4D} [(\xi_\alpha | \xi_\alpha) + \tau^2(\dot{\xi}_\alpha | \dot{\xi}_\alpha)] \right\} \tag{18}$$

we can derive the corresponding action $[D_{\alpha\beta} = D\delta_{\alpha\beta}]$

$$S_c(\varphi, B, \bar{\psi}, \psi) = S(\varphi, B, \bar{\psi}, \psi) - \tau^2 D \left(B_\alpha \left| \frac{\omega^2}{1+\omega^2\tau^2} B_\alpha \right. \right)_\omega \tag{19}$$

where S given by (9b) and $(a_\alpha|b_\alpha)_\omega$ means $\int(\mathrm{d}\omega/2\pi)a_\alpha(\omega)b_\alpha(-\omega)$. It is obvious that S_c is not invariant under the supersymmetric transformation (12) already for a potential system.

We can also perform the integration over the Bose fields $B_\alpha(t)$ and find an action $S(\varphi, \bar{\psi}, \psi)$ (see (21)). The corresponding Lagrangian includes additional terms with second derivatives of $\varphi_\alpha(t)$. These terms disappear in the white noise limit $\tau\to 0$.

Let us consider the simple special case

$$K_\alpha = -\Gamma\varphi_\alpha. \tag{20}$$

We find

$$S[\varphi, \bar{\psi}, \psi] = \frac{1}{4D}(\varphi_\alpha|(\omega^2+\Gamma^2)(1+\tau^2\omega^2)\varphi_\alpha)_\omega + (\bar{\psi}_\alpha|(\mathrm{i}\omega+\Gamma)\psi_\alpha)_\omega. \tag{21}$$

From here we conclude

$$C_{\alpha\beta}(\omega) = \langle\varphi(\omega)\varphi_\beta(-\omega)\rangle = \delta_{\alpha\beta}\frac{2D}{(\omega^2+\Gamma^2)(1+\omega^2\tau^2)} \tag{22}$$

and

$$G_{\alpha\beta}(\omega) = \langle\psi_\alpha(\omega)\bar{\psi}_\beta(\omega)\rangle = \frac{\delta_{\alpha\beta}}{-\mathrm{i}\omega+\Gamma}. \tag{23}$$

The FDT for the diagonal part of C and G reads

$$C(\omega) = \frac{2D}{\omega}\,\mathrm{Im}\,G\frac{1}{1+\omega^2\tau^2} \tag{24}$$

which can be also derived directly by solving the corresponding stochastic differential equation. For $\tau=0$ the usual form of the FDT (3) is obtained.

In conclusion, we have studied a dynamical system driven by a Gaussian white noise which does not fulfil detailed balance. The equivalence is demonstrated to the breaking of supersymmetric transformation. As a consequence we find that the response to an external field is not linearly related to the dissipation of the system manifest in a nonlinear FDT (17). The influence to the spin-glass phase is demonstrated elsewhere [15].

If the stochastic process is driven by coloured noise the field theoretic formulation leads to an action which is also not invariant under supersymmetry. This fact gives rise to deviations from the usual FDT (24).

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